Cognition Based Assessment in Elementary Mathematics: Student and Teacher Learning

Michael T. Battista
Michigan State University
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The Cognition-Based Assessment (CBA) project has developed a system that can be used to formatively assess and support students in constructing understanding and mastery of core ideas in elementary school mathematics.
CBA relevant for:

- Helping a Child Learn
- Classroom Instruction
- Designing Curricula
- Assessing Student Learning (F+S)

Requested by Teachers
Preliminary Remark 1 (Sort of): Learning
Constructivist View
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“Children do not see the world as we see it. Although they may observe the same phenomena we observe, they interpret them differently. ... From infancy to the grave, people construct their own views of reality, views that may not always coincide with what some would call objective reality. ... [E]verything we see is an interpretation.... What we 'see' may not be what is actually 'out there,' but rather the result of consistent misinterpretations.”

Alan Schoenfeld
Preliminary Remark 1 (Sort of): Learning Constructivist View

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Alan Schoenfeld

“I try not be intimidated by reality.”

Lily Tomlin

(sketch on the “Crazy Man”)
Preliminary Remark 2: 

The Nature of Learning

At every point in the process of a student’s construction of meaning for a mathematical idea, the nature of what gets constructed depends on the student’s current ideas and ways of reasoning.

Thus:

To guide and support students’ construction of meaningful mathematical ideas and reasoning we must understand how they are constructing meaning for particular mathematical ideas.
Teachers must "have an understanding of the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes"

(Carpenter & Fennema, 1991, p. 11).

(See also An, Kulm, Wu, 2004; Carpenter & Fennema, 1991; Clarke & Clarke, 2004; Fennema & Franke, 1992; Saxe et al., 2001; Schifter, 1998; Tirosh, 2000).
Research shows that such teacher knowledge can improve students' learning.  
(Fennema & Franke, 1992; Fennema et al., 1996)

Fennema et al. (1996) state, research provides "a convincing argument that one major way to improve mathematics instruction and [student] learning is to help teachers understand the mathematical thought processes of their students" (p. 432).

"There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students' changing conceptions as instruction proceeds" (Bransford et al., 1999, p. 11).

Research shows that improvements in formative assessment can produce significant learning gains (Black & Wiliam, 1998). CBA is a powerful type of formative assessment.
KEY IDEA in CBA

Levels of Sophistication

Researchers in mathematics education have found that students' development of conceptualizations and reasoning about numerous particular topics can be characterized in terms of "levels of sophistication."

These levels lie at the heart of CBA.
Levels of Sophistication

A) Start with the informal, pre-instructional reasoning typically possessed by students

B) End with the formal mathematical concepts targeted by instruction

C) Indicate cognitive plateaus students reach in moving from A to B
Levels of Sophistication Map Out the Cognitive Terrain for Learning Trajectories/Progressions
CBA Levels Are NOT LEVELS OF SOPHisticATION OF STUDENTS

Levels for reasoning.
Can vary by problem type.
Examine configurations of reasoning.
Once a student’s reasoning about an idea has been located in a “levels-of-sophistication” map, not only has the student's learning progress been pinpointed, but we have a good idea of where the student should proceed next to further develop that reasoning.
This “locating function”—which is the major source of power in cognition-based assessment—results not only from good assessment items, but from conceptual frameworks for interpreting students' reasoning on the items.
The jumps in the ascending plateau structure of a levels-model represent conceptual reorganizations that cause observable increases in sophistication in students' reasoning.
An ideal levels-of-sophistication model for a topic indicates jumps in sophistication that are small enough to fall within students "zones of construction."

A student should be able to accomplish the jump from Level N to Level N+1 while working to solve an appropriate set of problems.
CBA
Content Strands
The CBA project has developed assessment tasks and conceptual frameworks for eight mathematical topic strands:

1. addition and subtraction of whole numbers
2. whole-number place value
3. whole-number multiplication and division
4. length
5. area
6. volume
7. geometric properties of 2d shapes
8. area decomposition.
Levels of Sophistication

For Length
Non-Measurement and Measurement Reasoning

*Non-measurement* reasoning involves visually examining and reasoning about linear extension along a path.

*Measurement* reasoning involves the use of numbers to make judgments about how many unit lengths fit along a path or object.
<table>
<thead>
<tr>
<th>Non-Measurement Levels</th>
<th>Measurement Levels</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>visual comparison; no numbers used</strong></td>
<td><strong>numbers indicate how many unit lengths in an object</strong></td>
</tr>
<tr>
<td>NM0. Holistic Visual Comparison</td>
<td>M0: Use of Numbers Unconnected to Unit-Length Iteration</td>
</tr>
<tr>
<td>NM1. Unsystematic Comparison of Length of Parts</td>
<td>M1: Incorrect Unit Iteration</td>
</tr>
<tr>
<td>NM2. Systematic Decomposing/Recomposing</td>
<td>M2. Correct Unit Iteration</td>
</tr>
<tr>
<td>2.1. Rearrangement of Pieces to Make New Paths that Can Be Directly Compared</td>
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<tr>
<td>2.2. One-to-One Matching of Pieces</td>
<td></td>
</tr>
<tr>
<td>NM3. Comparison by Piecewise Property-Based Transformations</td>
<td>M3: Operating on Iterations</td>
</tr>
<tr>
<td></td>
<td>M4: Operating on Numerical Measurements</td>
</tr>
</tbody>
</table>
Non-Measurement Reasoning

N0. Holistic Visual Comparison

Focus on how objects look, with no systematic attention to parts. Imprecise, vague; length often confounded with other ideas such as time.

If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same?

S1: The top path is longer because "it wastes its time going up that way" [motioning to the two diagonal segments in the top path], and the bottom path "just goes straight."

S2: After drawing segments joining the left endpoints and the right endpoints of the two paths, "I think they are pretty much the same."
Non-Measurement Reasoning

N1. Unsystematic Comparison of Length of Parts

Students focus on comparing lengths of parts in vague, visual, and unsystematic ways.

*If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same? Why?*

[Diagram of two paths]

HT (grade 5) said that the second path would be longer, "cause like when it gets to these parts here [tracing along the first square indentation in the bottom path] it has bigger squares, so it would take longer to get through."
Non-Measurement Reasoning

N2. Systematic Decomposing/Recomposing

2.1. Rearrangement of Path Pieces to Make New Paths that Can Be Directly Compared

On top path, S drew one segment at the right end and another under the triangular indentation, saying, "One of them will make this and one of them will be at the bottom. And it [the top path] will be a little bit bigger by one line."

2.2. One-to-One Matching of Pieces

S claimed that the 2 wires are the same. She matched segments in the bottom wire with segments in the top wire, saying for each, "This is the same as this."
M0. Use of Numbers Unconnected to Unit-Length Iteration

Use counting that does not represent the iteration of a fixed unit length.

When asked how she knew the rectangle she drew was 40, JAK (grade 2) drew irregularly spaced dots along its inside edge, stopping when her count reached 40. There was no indication that JAK was using dots as indicators of fixed units lengths.
Measurement Reasoning

M1. Incorrect Unit Iteration
Iterations of unit lengths are incorrect because (a) S do not know what a length unit is, or (b) they fail to properly coordinate units —gaps, overlaps, or different length units.

SAS (grade 4) counted squares for the gray path as shown. —>
M2. Correct Unit Iteration
Units properly maintained, coordinated, and located so that gaps, overlaps, and variations in units are eliminated.

How many rods does it take to cover around the gray rectangle [pointing around perimeter]?
M3. Operating on Iterations
Determine measurements without iterating every unit.

How many black rods does it take to cover around the gray rectangle?

BW (grade 5) counted the spaces between the hash marks on the top side, getting 5 rods. He said that since the bottom was the same as the top, it would also be 5. He then counted 3 rods on the left side. He said that the left side was equal to the right side, so the right side also would be 3. BW then said, "3+3=6 and 5+5=10. So it takes 16 black rods."
Measurement Reasoning

M4. Operating on Numerical Measurements
Numerically or inferentially operate on length measurements without iterating unit lengths. Integrate and apply the processes from Non-Measurement 3 with measurement reasoning.

JO (grade 8) labels the missing sides as shown. When asked how he got 3 and 4 for the missing lengths of the vertical sides, JO said, "This [the side he labeled 4] would be greater [than the other unlabeled vertical side], so that'd be 4 and 3 because 4 and 3 is 7." When asked about the 5 and 3, JO said, "Well, 8 is the whole length [goes across the top side] of the bottom [goes across what would be the entire bottom]. You take that [points at the horizontal segment labeled 3] away from 8 [points to the top side] and it'd equal 5 [points to the side he labeled 5]."
Overview of Numerical Data for Length
CBA Tasks:
Non-Measurement versus Measurement Reasoning

There were 5 types of tasks that did not specify that measurement should be used and were also used in all 3 years of testing.
## Percent of Responses at Each Grade

<table>
<thead>
<tr>
<th>Level/Grade</th>
<th>N0</th>
<th>N1</th>
<th>N2.1</th>
<th>N2.2</th>
<th>N3</th>
<th>M0</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>N total</th>
<th>M total</th>
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<tr>
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<td></td>
<td></td>
<td></td>
<td>11.4</td>
<td>2.9</td>
<td></td>
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<td>85.7</td>
<td>14.3</td>
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<td>10.0</td>
<td>16.7</td>
<td>1.7</td>
<td>15.0</td>
<td>3.3</td>
<td>1.7</td>
<td></td>
<td></td>
<td></td>
<td>80.0</td>
<td>20.0</td>
</tr>
<tr>
<td>3rd</td>
<td>29.5</td>
<td>6.4</td>
<td>12.8</td>
<td>2.6</td>
<td>14.1</td>
<td>19.2</td>
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<td></td>
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<td>48.7</td>
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<td>4.4</td>
<td>11.8</td>
<td>5.9</td>
<td>13.2</td>
<td>10.3</td>
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<td>51.5</td>
<td>48.5</td>
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<tr>
<td>5th</td>
<td>33.7</td>
<td>10.5</td>
<td>15.1</td>
<td>2.3</td>
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<td>2.8</td>
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<td>13.5</td>
<td></td>
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<td></td>
<td>63.0</td>
<td>37.0</td>
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</table>
CBA2
An Investigation of Elementary Teachers' Learning, Understanding, and Use of Research-Based Knowledge about Students' Mathematical Thinking

NSF ESI-0554470
TPC Research Study
Start Date: June 1, 2007
Major Question for CBA2 Project

How do pre- and in-service teachers construct understanding of and pedagogical relevance for research-based knowledge on students' mathematical thinking before, during, and after instruction on that thinking, and what factors affect this construction of knowledge?
Despite the importance of such knowledge, Carpenter, Fennema, Franke, and colleagues found that elementary school teachers' knowledge of students' thinking about addition and subtraction of whole numbers was fragmented and not organized in ways that enabled them to understand students' thinking or apply that knowledge to instruction.
Phase 1/Year 1
Individual Teacher Case Studies

Phases 2/3
Classroom Teaching Experiments
Phase 1

20 Teachers Participated in Summer Workshops on CBA

14 Case Studies During Academic Year, 5 from Workshop
Phase 1

PRELIMINARY

Data and Questions

(from Workshops, Writing, Interviews)
How Do Teachers Think About Using CBA Ideas?
Assessment of Student Understanding
T: I can use the ideas in the CBA document in my teaching by using assessment tasks appropriate to my students’ needs, at the beginning of the unit, throughout the unit and as a final assessment piece at the end of the unit to assess the level of student understanding about length.
Diagnosis and Remediation

(Including Analyzing Student Misconceptions)
T: I think it [CBA document on length] would help to better analyze student’s misconceptions and find ways to guide them to better understanding.
Assessment of Understanding For The Purpose of Guiding Instruction
T: It would be interesting to use the tasks in the Length document to pretest the students, then depending on their reasoning abilities, work with them to get them to the highest levels of reasoning.
Understand Students’ Mathematical Thinking, Strategies, Needs, Difficulties
T: The ideas in the document helped me to understand the various levels of mathematical thinking. I will be able to look for these strategies that my students use in the classroom.
Use Specifically (Exclusively) With Students Having Difficulty With Length
T: I would use the *Levels of Sophistication in Students Reasoning about Length* specifically with my students that were having problems with length.
Understanding Levels Helps With Reflecting On Or Redesigning Instruction
T: I’ve noticed that my thinking about students’ abilities has changed from my previous, “gets it or doesn’t get it” mentality. … I feel we need to dig deeper into fewer topics so students have the ability to reach the upper levels of CBA.
T: I found out that some of the math games I was using were not in the best interest of the students. I was teaching rote memorization instead of number concepts.
T: After reading this [the CBA document on length] I thought, “Oh those poor children, that was a terrible measurement unit that I did.” I feel I didn’t have a lot of reasoning behind my measurement unit this year, as far as length.
T: Thinking back to my [previous] assessments … are all very tool and unit based. Whereas none of these [CBA] assessments were. So I was like scrapping everything. Because it wasn’t what I saw the standards defining measurement as. I’m going to redo the way I teach length or the concept of measurement.
Communicate Students’ Development To Parents
T: Using this document and giving pretests could help me understand the individual needs of the students and when communicating with parents it would give them a better picture of their child’s development and understanding of measurement.
Ideas For Specific Instructional Tasks
T: I really liked the tasks given throughout the document, as well as the responses from the students. I love to read or hear what the students are thinking and how they figured something out. They come up with ways I never thought of and spark great discussions then in class with other students.
Thinking About State Tests
T: I have felt for years that we rush children in their learning to satisfy Ohio assessments. The workshops reinforced my views. We present concepts, but many students don’t truly learn them.
T: The [CBA] documents reinforced my belief that the Ohio Content Standards are mis-mapped across grade-levels and not always developmentally appropriate. Further, Ohio assessments inconsistently and/or ineffectively assess learned content.
What Difficulties Do Teachers Initially Have with Using CBA?
T: I think it would be awesome to sit down with each child, give them various tasks, and watch/learn where their reasoning at length/measurement is. But our curriculum is so packed, I don’t know where I would find the time. Plus, what would the other students do?
T: I plan on studying the different levels of learning individually and seeing if I can observe them in my students thinking. When I feel proficient in doing that, I look forward to using that information in planning my instruction.
T: I would like to work with other teachers in mapping out a plan for using the tasks as individual assessments in the classroom (and how to apply that data to instruction), but I am unsure of the logistics in carrying this out in a classroom of twenty-seven students. For now, I plan to use some of the tasks from both documents in “problem of the day” type activities.
T: Reading the materials only once, I don’t have a strong enough understanding of how exactly this should change my instruction, but I do see changes should be made.
T: I would readily use the tasks provided in the document. Not only because they are valuable in assessing students’ understanding, but also because the document in its entirety would allow me to interpret my students’ performance on the tasks. I doubt I would be able to utilize alternate tasks and assess the outcomes in the same manner as the document because I do not wholly understand it as immediately applicable in the classroom.
T: I guess I’m thinking, I should be doing a much better job than I’m doing. … So I’m wondering if maybe I shouldn’t be doing so much of the rote. Maybe that’s hurting instead of helping. I don’t know. I’m so confused.
Differences in Thinking about Length Assessment before and after Reading the CBA Length Document
Task

Describe the kinds of assessment tasks you would give your students to determine if they understand the concept of length.
Q1: “The students should be able to accurately measure the length of various objects with the appropriate measurement tools [e.g., meter stick, not ruler]. Some of the objects might include finding the length of a rectangle, the length of the classroom, the length of the classroom door etc. The students would also be expected to use an appropriate measurement tool for each assigned task.”
In Q1, T1 focuses on students accurately measuring using a ruler and selecting an appropriate measurement “tool.” She focuses on a specific procedure for measuring. She refers to selecting an appropriate “tool” rather than an appropriate unit of measure, the latter seeming more conceptual. She only describes one type of task, measuring the lengths of different objects. She does not refer to measurement reasoning.
Q2: “If we are just beginning our study of length, I might give the students a task to determine their level of understanding of length. The task might be the Task A4 … where the students decide which path is shorter, longer or the same length. I would ask the students to explain their reasoning and thinking, so that I might obtain a better idea of how well they understand the concept of length. This would then give me a starting point with each student and can be used to design instruction and tasks throughout the remainder of the unit.”
In Q2, in addition to changing the type of task she would use, T1 talks about giving a length assessment at the beginning of the length unit to determine students’ “level of understanding of length.” She wants students to explain their reasoning so that she understands how well her students understand the concept of length, which would give her a starting point for designing instruction. T1’s focus seems to shift from a measurement procedure to measurement reasoning. T1’s thinking about length assessment seems to shift from summative to formative, with her saying explicitly that her instruction needs to be based on understanding her students’ thinking about length.
Describe the different kinds of strategies or reasoning you think your students would use on your assessment tasks. Include both correct and incorrect strategies and reasoning.

T3: If they were just using the piece of yarn…they might not stretch it out fully, they might wrinkle it up, and then I’d have to show them that you have to give it the full stretch, the full length….If they were measuring and they did not have more than one meter stick, a lot of times they’ll start at a point and kind of approximate where it ends, but they won’t put a mark there and begin with the mark again. So they might have a 4 inch gap or some centimeter gap or whatever. Or sometimes if they have two or three meter sticks, they won’t butt them up against each other, so you’d have to tell them they have to touch, you can’t just leave a big gap and say that you’ve measured accurately.
A Deeper Look at Teachers’ Understanding of the CBA Length Document
Episode of a teacher thoroughly engaged with trying to make sense of a student’s thinking about length—one of the goals of CBA.

Although this teacher’s analysis of the student’s thinking might not be all that we would like, her reflection on student thinking is an important first step. We must understand teachers’ sense making and learn how to build on it.
Task

How many black rods does it take to cover the gray rod? (No concrete rod given.)
Task A8. How many black rods does it take to cover the gray rod?

SA (grade 2) said that she knew that the black rod "takes 3 hash marks" on the gray segment. She drew a vertical segment from the right end of the black rod to the third hash mark on the gray segment. Moving from left to right, SA then counted the fourth, fifth, and sixth hash marks, "1, 2, 3," marked the sixth hash mark, and said, "have one." She counted, "1, 2, 3" on the seventh, eighth, and ninth hash marks and said "have one." She returned to the beginning of the gray rod, pointed to each section she created, and counted "1, 2, 3."

In this episode, SA made a widespread student error. Initially, she focused on the unit length (the black segment), noting that 3 hash marks occurred under it. However, as she counted hash marks to indicate iterations of this unit, she lost track of the length unit and focused only on the hash marks. The connection between the hash marks and unit length was lost.
SA’s Reasoning

Takes 3
What T4 Said

T4: One thing that I wrote about that I disagreed kind of, where she started with the two unit and she counted two units [dragging her pen along the two hash-mark-to-hash-mark segments below the black rod] and then she counted three units and three units [dragging her pen along the next two groups of three hash-mark-to-hash-mark segments] and they said that she lost track of the length unit.

I don't think she did. I think she went one, two, three [pointing at hash marks under the rod], and then she had the three in her head, and then she went one, two, three, one, two, three [pointing at successive groups of three hash-mark-to-hash-mark segments].

Commentary: T4 seems to disregard the description that SA pointed only at hash marks—she did not point at segments. But T4’s comment that SA counted 1, 2, 3 and had that count sequence in her head as she counted 1, 2, 3 twice more is important and consistent with the CBA interpretation.
T4: Another explanation is that they began counting on the first hash mark to get three, and then counted on correctly after that point, three hash marks. This is a common occurrence in first and second grade. Just as with addition and counting on, low level students tend to begin with the number they are adding on to, instead of the next number. The same frequently occurs in measurement. They begin measuring at the 1 inch mark instead of at the beginning of the ruler. They often make this mistake when using a number line.

Commentary: First, T4’s claim that SA “counted on correctly” after her first count of three is correct for counting hash marks, but incorrect for delineating unit lengths. SA’s second and third counts of three did not delineate linear units the same size as the black rod.
Takes 3 1 2 3 1 2 3
Overall Commentary

How should we interpret T4’s analysis of SA’s reasoning?

First, T4 interprets SA’s actions in a way that is not consistent with the data. While counting, SA pointed only at hash marks, never length “units.” (T4 seems to “see” unit lengths in SA’s gestures toward hash marks, which is typical and almost automatic for “knowing” adults.)

Second, even in T4’s interpretation, SA does not establish and maintain the required unit length (the black rod). But T4 does not seem to attend to this mistake; she attends more to the procedure of counting than to unit length iteration, which is consistent with our emerging finding that some teachers focus on the procedures rather than conceptual aspects of length measurement. T4’s focus on counting may also occur because she sees SA’s mistake as similar to a mistake that students make in the counting-on procedure.
A major question that we must ask is if the conceptual depth in T4’s analysis is sufficient to understand SA’s reasoning in a way that can help T4 remediate SA’s difficulty or alter her overall instruction on length.

If SA had been T4’s student, or if T4 changes her length instruction as a result of this interpretation, T4’s focus on counting rather than proper iteration of the unit length might cause her subsequent instruction to miss the mark.

However, much of teaching is about making adjustments. So even if T4’s original changes to instruction miss the mark, if she continues to try to make sense of students’ thinking, she might redirect appropriately.
Questions
How can we get teachers to deepen their understanding of CBA concepts?

LINKED WITH

How can we help teachers use CBA concepts in their teaching?
The project is now moving from investigating teachers’ understanding of CBA to teaching experiments meant to increase their understanding and use of CBA (along with our understanding of teachers’ learning).
What level of detail or depth of analysis of student thinking (CBA levels) are teachers “capable of” (or likely to achieve) within the context of their professional lives?
Tension

Simplifying CBA levels makes them more understandable, but less powerful in analyzing students’ thinking.
END